Correction Ex. 1 et 3

1) 1). $u_x + u_y + u_z = u$

Changement de coordonnées:

$$\begin{cases} \tilde{x} = x+y+z \\ \tilde{y} = x-y \\ \tilde{z} = y-z \end{cases} \Longrightarrow \begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \tilde{x}} + \frac{\partial}{\partial \tilde{y}} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial \tilde{x}} - \frac{\partial}{\partial \tilde{y}} + \frac{\partial}{\partial \tilde{z}} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial \tilde{x}} - \frac{\partial}{\partial \tilde{z}} \end{aligned}$$

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = 3\frac{\partial}{\partial \tilde{x}}$$

L'équation devient

$$3u_{\tilde{x}} = u \Rightarrow u = e^{\tilde{x}/3} + \underset{\substack{\uparrow \\ \text{fonction arbitraire}}}{F}(x-y, y-z)$$

d'où la solution générale

$$u(x,y,z) = e^{\frac{x+y+z}{3}} + F(x-y, y-z).$$

2). $u_t = u u_x$

$$\partial_t F(x+ut) = F'(x+ut)\,[u + u_t t]$$

$$\partial_x F(x+ut) = F'(x+ut)\,[1 + u_x t]$$

Donc si $u = F(x+ut)$, on a

$$u_t - u u_x = F'(x+ut)\,[\cancel{u} + u_t t - \cancel{u} - u u_x t] =$$

$$= t F'(x+ut)\,[u_t - u u_x]$$

ce qui implique $u_t = u u_x$.

Cela signifie que la solution générale est implicitement donnée par l'équation $u(x,t) = F(x + t u(x,t))$, où F est une fonction arbitraire.

3] $u_{tt} - u_{xx} = -2gu(a^2 - u^2)$

Si on pose $u(x,t) = f(x - vt)$, alors

$$u_t = f'(x - vt) \cdot (-v)$$

$$u_x = f'(x - vt)$$

$$u_{tt} = f''(x - vt) \cdot v^2$$

$$u_{xx} = f''(x - vt)$$

et donc f vérifie une éq. diff. ordinaire :

$$f''(v^2 - 1) = -2g f (a^2 - f^2)$$

$$\Rightarrow \quad f'' = -2\frac{g}{v^2-1} f(a^2 - f^2) \qquad \Big| \cdot 2f'$$

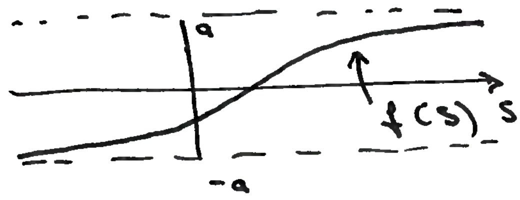
$$2f'f'' = -4\frac{g}{v^2-1} f(a^2 - f^2) f'$$

$$(f'^2)' = -\frac{g^2}{v^2-1}(2a^2f^2 - f^4)' =$$

$$\Rightarrow \quad = +\frac{g^2}{v^2-1}\left[(f^2 - a^2)^2\right]'$$

$$(f')^2 = \frac{g^2}{v^2-1}(f^2 - a^2)^2 + \text{const}$$

Les conditions $f(\pm\infty) = \pm a$, $f'(\pm\infty) = 0$ impliquent que const $= 0$, et donc



$$f' = \pm\frac{g}{\sqrt{v^2-1}}(a^2 - f^2)$$

Cette EDO est séparable :

$$\frac{df}{a^2 - f^2} = \pm\frac{g}{\sqrt{v^2-1}}\,ds \qquad \Big| \int$$

$$\int \frac{df}{a^2 - f^2} = \frac{1}{2a}\int\left(\frac{1}{a-f} + \frac{1}{a+f}\right)df = \frac{1}{2a}\ln\frac{a+f}{a-f}$$

$$= \pm\frac{g}{\sqrt{v^2-1}}(s - s_0)$$

$$\Rightarrow \quad \frac{a+f}{a-f} = e^{\pm\frac{2ga}{\sqrt{v^2-1}}(s - s_0)} \Longrightarrow f(s) = a\tanh\frac{ga(s - s_0)}{\sqrt{v^2-1}}$$